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Hadronic light-by-light scattering contribution to the muon magnetic anomaly: constituent quark loops and QCD effects

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The hadronic light-by-light scattering contribution to the muon anomalous magnetic moment can be estimated by computing constituent quark loops. Such an estimate is very sensitive to the numerical values of the constituent quark masses. These can be fixed by computing the hadronic vacuum polarization contribution to the muon magnetic anomaly within the same model. In this Letter, we demonstrate the stability of this framework against first-order perturbative QCD corrections.

The measurement of the muon anomalous magnetic moment completed about seven years ago [1] resulted in one of the most interesting controversies in contemporary high-energy physics. When the very precise experimental measurement

$$a_\mu^{\rm exp} = 11\ 659\ 2080(63) \times 10^{-11}, \tag{1}$$

is compared to the highly-developed theoretical value [2]

$$a_\mu^{\rm th} = 11\ 659\ 1790(65) \times 10^{-11}, \tag{2}$$

a 3.2σ discrepancy is observed:

$$a_\mu^{\rm exp} - a_\mu^{\rm th} = (290 \pm 90) \times 10^{-11}. \tag{3}$$

The size of the discrepancy is tantalizingly close to that predicted by the most popular extensions of the Standard Model of particle physics, such as Supersymmetry [3].

The Standard Model value of the muon anomalous magnetic moment is obtained by combining QED, electroweak and hadronic contributions. While the QED and electroweak corrections to the muon anomalous magnetic moment are well-established, the hadronic contributions are notoriously difficult to handle. Indeed, given the small value of the muon mass $m_\mu = 105.7$ MeV, these contributions are sensitive to non-perturbative hadron physics. Their calculation from first principles is nearly impossible except, perhaps, by lattice field theory methods [4, 5].

It is useful to distinguish two types of hadronic contributions to the muon magnetic anomaly – the hadronic vacuum polarization and the hadronic light-by-light scattering. The calculation of the hadronic vacuum polarization contribution uses dispersion relations to connect it to the integral of the annihilation cross-section $\sigma(e^+e^- \to \text{hadrons})$ over center-of-mass energies. This cross-section has been measured over a wide range of energies and is known precisely. The hadronic vacuum polarization contribution to the muon anomalous magnetic moment can therefore be calculated in an essentially model-independent way [6, 7].

On the contrary, the hadronic light-by-light scattering contribution to the muon anomalous magnetic moment can not be directly related to any experimental data. Its calculation is therefore bound to rely on model assumptions. Popular models that address the computation of the hadronic light-by-light scattering contributions utilize the existence of two parameters in QCD – the number of colors N_c and the mass of the pion m_π. It is well-known that in the limit of large N_c or small m_π, low-energy QCD simplifies. In the chiral $m_\pi \to 0$ limit, the dominant contribution to the muon anomalous magnetic moment comes from loops of charged pions whose interaction with photons is given by scalar QED, up to power corrections. In the large-N_c limit, the dominant contribution comes from approximating the photon-photon scattering by single-meson exchanges.

It is a feature of all existing model calculations of the hadronic light-by-light scattering contribution [8–12] that the chiral enhancement is not effective. A plausible explanation of this feature is given in Ref. [12], where the anatomy of the charged pion loop contribution, including the power-suppressed terms, is studied. The absence of a meaningful chiral enhancement leaves us with the number of colors N_c as the only parameter to employ. Interestingly, the dominance of this parameter allows us to extrapolate the calculation from the low-momentum region to high-momentum region since the quark contribution to hadronic light-by-light scattering is enhanced in the large-N_c limit. As pointed out in Ref. [12], this feature leads to a useful QCD constraint on the hadronic light-by-light scattering amplitude.

It is possible to take these ideas to the extreme and

estimate the *full* hadronic light-by-light scattering contribution to the muon anomalous magnetic moment from *constituent* quark diagrams alone. This approach has a long history. Its early stages are described in Ref. [13]. The quark loop computation was used to question the sign of $a_\mu^{\rm hlbl}$ in Refs. [14, 15] and, gradually, it started becoming presented [15, 16] as a respectable way to estimate the hadronic light-by-light scattering contribution to the muon magnetic anomaly.

There are two immediate problems that a computation of the hadronic light-by-light scattering through the constituent quark loop must address. The first problem is that $a_\mu^{\rm hlbl}$ depends strongly on the constituent quark masses. However, those masses can be estimated by requiring that *the same model* correctly predicts the hadronic vacuum polarization contribution to the muon anomalous magnetic moment. Since the latter can be also obtained in a model-independent way by integrating the annihilation cross-section $\sigma(e^+e^- \to {\rm hadrons})$, we get a very useful constraint. Such a procedure was suggested in Ref. [15] and later adopted in Ref. [16]. In both of these references, subtle details including $SU(3)$-flavor mass differences of constituent quarks were included. It is interesting that this approximate framework appears to be remarkably consistent with other computations based on different models for photon-hadron interactions. For example, in Ref. [16] the hadronic light-by-light scattering contribution was estimated to be $a_\mu^{\rm hlbl} = (137^{+25}_{-15}) \times 10^{-11}$, in *nearly perfect* agreement with the result of Ref. [12]. A more recent estimate of the hadronic light-by-light scattering contribution $a_\mu^{\rm hlbl} = (106 \pm 25) \times 10^{-11}$ was given in Ref. [17]. It attempts to accommodate many existing estimates of this quantity in a self-consistent manner, and is consistent with the results of Refs. [15, 16].

The second problem of the constituent quark model is that it does not possess the correct chiral limit of QCD unless quarks directly couple to the Goldstone bosons, *i. e.* the pions. Such extensions of the constituent quark model are known and are well-documented [20–22]. In the context of the hadronic-light-by-light scattering contribution to the muon magnetic anomaly, the π^0 couples to photons through a constituent quark loop. This introduces a double logarithmic contribution to the muon magnetic anomaly, $a_\mu^{\rm hlbl} \sim \ln^2 M_q/m_\pi$ [10, 23]. This chiral logarithm cannot be reproduced by a loop of constituent quarks that couple to photons only. However, empirical evidence suggests that the leading logarithmic contribution is not numerically important; for example see Ref. [24]. Moreover, the logarithmic enhancement becomes ineffective if the numerical values of quark masses become comparable to the physical pion mass. As we will see in what follows, the quark masses obtained from fits to hadronic vacuum polarization are relatively small. This provides justification for neglecting diagrams with the π_0 exchanges. The largest contribution is then due to the constituent quark loop. We henceforth focus on this term.

Before discussing the constituent quark loop in detail, we point out that the good agreement between results for $a_\mu^{\rm hlbl}$ obtained by different groups over many years was recently challenged by the results presented in Ref. [18]. In that reference, a computation based on the Dyson-Schwinger equation in QCD lead to a result for the hadronic light-by-light scattering larger than other computations by about a factor of two. The authors of Ref. [18] attributed their result to large modifications of the quark-photon vertex, when this vertex is computed dynamically in nearly full QCD.

It is important to understand and possibly cross-check the results reported in Ref. [18]. Regardless of the outcome, a resolution of this problem will increase our confidence in the computation of the hadronic light-by-light scattering contribution to a_μ. The need for that is quite urgent, given plans to continue the $g-2$ experiment at Fermi National Accelerator Laboratory, and to reduce the experimental error in $a_\mu^{\rm exp}$ down to 20×10^{-11} [19]. Such an experimental precision approaches the uncertainty in many existing estimates of the hadronic light-by-light scattering contribution. It is definitely *much smaller* than the difference between central values in $a_\mu^{\rm hlbl}$ quoted in Ref. [17] and Ref. [18].

In this Letter, we take an initial step towards a better understanding of the hadronic light-by-light scattering contribution to the muon magnetic anomaly by calculating the QCD radiative correction to the constituent quark loop. Since the momentum scales in this problem are so low, one may wonder whether or not computation of QCD corrections is at all meaningful. The point of view that we will adopt here is that a constituent quark model with dynamical gluons and pions is a consistent field theoretical framework to describe QCD phenomena at moderate energies [22]. As we explained above, there are good reasons to expect the constituent quark loop to dominate. Studying constituent quark loop diagrams and QCD corrections to it is also interesting for the following reason. Unlike Refs. [15, 16] this model has *two* parameters – the constituent quarks mass *and* the value of the strong coupling constant at low energies. This allows us to test if QCD dynamics affects $a_\mu^{\rm hlbl}$ and $a_\mu^{\rm hvp}$ in a similar or totally different way. In particular, QCD radiative corrections directly check the suggestion of Ref. [18] that large modifications of the photon-quark vertex may occur in the hadronic light-by-light scattering contribution, but can remain undetected in the hadronic vacuum polarization. When calculations of the hadronic vacuum polarization and the hadronic light-by-light are truncated at the same order in the $\mathcal{O}(\alpha_s)$ expansion, the model gives *a unique* predictive relation between the two quantities.

Given the approximate nature of our computation, we assume that constituent quark masses are significantly larger than the mass of the muon and we work to leading order in m_μ/M_q [32]. We completely neglect the $SU(3)$ mass splittings and we use $M_u = M_d = M_s = M_q$. In such an approximation, the hadronic vacuum polariza-

tion contribution and the hadronic light-by-light scattering contribution read [25, 26]

$$
\begin{aligned}
a_\mu^{\rm hvp} &= \left(\frac{\alpha}{\pi}\right)^2 \frac{N_c m_\mu^2 \langle Q_q^2\rangle}{45 M_q^2}, \\
a_\mu^{\rm hlbl} &= \left(\frac{\alpha}{\pi}\right)^3 \left(\frac{3}{2}\zeta(3) - \frac{19}{16}\right) \frac{N_c \langle Q_q^4\rangle m_\mu^2}{M_q^2}.
\end{aligned}
\tag{4}
$$

In Eq. (4), $\langle Q_q^n\rangle$ with $n = 2, 4$ denote sums over the electric charges of u, d and s quarks in the appropriate powers. Taking the ratio of two contributions in Eq. (4), we find

$$
a_\mu^{\rm hlbl} = a_\mu^{\rm hvp} \, \frac{\alpha}{\pi} \left(\frac{3}{2}\zeta(3) - \frac{19}{16}\right) \frac{45\langle Q_q^4\rangle}{\langle Q_q^2\rangle}. \tag{5}
$$

We use $a_\mu^{\rm hvp} = 6900 \times 10^{-11}$ and find $a_\mu^{\rm hlbl} = 148 \times 10^{-11}$, in reasonable agreement with previous estimates [17].

We point out that these numerical results for the hadronic vacuum polarization and the hadronic light-by-light scattering contributions require small quark mass values, $M_q \approx 200$ MeV. This makes the physical interpretation of the parameter M_q somewhat obscure. Indeed, it is well understood that the ρ-meson with the mass $M_\rho \sim 770$ MeV gives the largest contribution to $a_\mu^{\rm hvp}$. Since the mass scale $2M_q \sim 400$ MeV is significantly smaller than M_ρ, the fact that one can fit the hadronic vacuum polarization contribution to a_μ with these mass parameters emphasizes how unphysical they are. In fact, the contribution of the ρ-meson to $a_\mu^{\rm hvp}$ is determined by two *independent* parameters – the mass of the ρ meson and the $\rho - \gamma$ mixing parameter $g_\rho \sim 5$. However, in the constituent quark approximation, the quark mass parameter M_q fits simultanesouly their combination. One finds $M_q \sim M_\rho/g_\rho$ which justifies the low value of the quark mass. An apparent relevance of this small mass scale for other observables, beyond hadronic vacuum polarization, suggests an existence of an infrared (below the ρ mass) duality between hadronic and quark contributions, whose origin and precise meaning remain unclear at the moment [27].

We would like to check the sensitivity of Eq.(5) to QCD radiative corrections. The $\mathcal{O}(\alpha_s)$ correction to the hadronic vacuum polarization contribution can be read of from Ref. [28]. We find

$$
a_\mu^{\rm hvp,NLO} = a_\mu^{\rm hvp} \left[1 + \frac{205}{54} C_F \left(\frac{\alpha_s}{\pi}\right)\right], \tag{6}
$$

where $C_F = 4/3$ is a Casimir invariant of the $SU(3)$ color group. It is clear from Eq. (6) that the QCD correction is significant. For numerical estimates, we take $\alpha_s = 0.35$, which corresponds to renormalization scales of about 1 GeV. We find that the QCD corrections to $a_\mu^{\rm hvp,NLO}$ amount to sixty percent. While this is a large correction, it is not entirely meaningful since we can redefine the numerical value of the quark mass M_q to absorb it, thereby avoiding any change in $a_\mu^{\rm hvp}$. However, it is important to check how such modifications of model parameters change the hadronic light-by-light scattering contribution. It is clear that if the hadronic light-by-light scattering contribution receives corrections that are similar to Eq. (6), the prediction shown in Eq. (5) is hardly affected.

To check if this is indeed the case, we must compute the QCD correction to $a_\mu^{\rm hlbl}$ in Eq.(4). Such a calculation involves sixty four-loop diagrams. Representative examples are shown in Fig. 1. In general, the computation of four-loop diagrams is very difficult, if not impossible. However, in our case it is relatively straightforward for two reasons. First, because we are interested in the computation of the anomalous magnetic moment, no momentum is transfered from the magnetic field line to the muon line, so that relevant diagrams can be reduced to self-energy diagrams. Second, since we are interested in the limit $M_q \gg m_\mu$, we can expand those diagrams in a Taylor series in m_μ and the incoming momentum of the muon line, reducing the self-energy diagrams to four-loop vacuum bubbles. Upon expanding, all loop propagators have momenta that are parametrically proportional to heavy quark masses. This momentum region gives the only contribution to the muon magnetic anomaly that scales as m_μ^2/M_q^2.

Four-loop vacuum bubble diagrams with a single mass parameter M_q are well studied in the literature. They can be evaluated using various implementations of the Laporta algorithm [29] for solving integration-by-parts identities [30]. The master four-loop vacuum bubble integrals that we require are also known and can be found in Ref. [31]. Performing all the necessary steps, we arrive at the following result

$$
a_\mu^{\rm hlbl,NLO} = a_\mu^{\rm hlbl,LO} \left(1 + \frac{\alpha_s}{\pi} C_F \frac{\Delta_1}{\Delta_0}\right), \tag{7}
$$

where $\Delta_0 = -19/16 + 3/2\zeta(3)$ and

$$
\begin{aligned}
\Delta_1 = &-\frac{473\pi^2}{1080}\ln^2 2 + \frac{52\pi^2}{405}\ln^3 2 - \frac{42853\pi^4}{259200} \\
&+ \frac{5771\pi^4}{32400}\ln 2 + \frac{473}{1080}\ln^4 2 - \frac{52}{675}\ln^5 2 - \frac{8477}{2700} \\
&+ \frac{473}{45}a_4 + \frac{416}{45}a_5 + \frac{34727\zeta_3}{2400} - \frac{23567\zeta_5}{1440}.
\end{aligned}
\tag{8}
$$

In Eq. (8) $a_{4,5} = {\rm Li}_{4,5}(1/2)$ are the values of polylogarithmic functions and $\zeta_{3,5}$ are the values of Riemann zeta-function at the respective argument. Numerically, $C_F\Delta_1/\Delta_0$ evaluates to 3.851.

We take the ratio of Eqs. (6,7), equate it to the ratio of physical quantities and obtain

$$
a_\mu^{\rm hlbl} = R^{\rm NLO}\, a_\mu^{\rm hvp}, \tag{9}
$$

where $R^{\rm NLO} = a_\mu^{\rm hlbl,NLO}/a_\mu^{\rm hvp,NLO}$. The ratio of NLO contributions reads

$$
R^{\rm NLO} = \; f\left(\frac{\alpha_s}{\pi}\right) \, \frac{\alpha}{\pi} \left(\frac{3}{2}\zeta_3 - \frac{19}{16}\right) \frac{45\langle Q_q^4\rangle}{\langle Q_q^2\rangle}. \tag{10}
$$

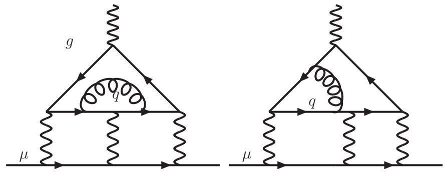

FIG. 1: Examples of diagrams that contribute to higher-order corrections to the hadronic light-by-light scattering contribution to the muon magnetic anomaly in the constituent quark model.

The effects of α_s corrections are absorbed into the function $f(\alpha_s)$, which reflects the *relative* sensitivity of the hadronic vacuum polarization and hadronic light-by-light contributions to radiative corrections. We find

$$f(x) = \frac{1+3.851\,x}{1+5.061\,x}. \tag{11}$$

This function depends only weakly on the strong coupling constant, changing from $f(0) = 1$, to $f(1) = 0.8$, to $f(\infty) = 0.76$. Taking $a_\mu^{\rm hvp} = 6900 \times 10^{-11}$ and α_s in the interval $[0,..,\pi]$, we find $a_\mu^{\rm hlbl}$ to be in the range $a_\mu = (118 - 148) \times 10^{-11}$, consistent with earlier model computations [17]. The QCD effects tend to slightly decrease the hadronic light-by-light scattering contributions to the muon magnetic anomaly relative to hadronic light-by-light scattering, since $f(\alpha_s) < 1$.

In summary, we studied the prediction of the constituent quark model for the hadronic light-by-light scattering contribution to the muon magnetic anomaly, including QCD radiative effects. While it is known that pions need to be explicitly introduced into the model to correctly describe the chiral limit of QCD, we argued that, for the muon magnetic anomaly, this issue is numerically not important. QCD radiative corrections to constituent quark loop diagrams provide a tool to diagnose the sensitivity of the vacuum polarization and the light-by-light scattering contributions to various aspects of QCD dynamics. We demonstrated that the ratio of hadronic vacuum polarization and hadronic light-by-light scattering contributions to the muon $g-2$ is remarkably stable against QCD radiative corrections. Within our approach, we are not able to detect a large renormalization of the quark-photon vertex – particular to hadronic light-by-light scattering contribution – that is claimed to be responsible for the large enhancement of $a_\mu^{\rm hlbl}$ observed in Ref. [18]. We believe that our result indicates that the constituent quark model computation of the hadronic light-by-light contribution to the muon magnetic anomaly is robust.

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